

The Cayley Plane and the Witten Genus

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ABSTRACT. This paper defines a new genus, the *Cayley plane genus*. By definition it is the universal multiplicative genus for oriented Cayley plane bundles. The main result (Theorem 2) is that it factors (tensor \mathbf{Q}) through the product of the Ochanine elliptic genus and the Witten genus—revealing a synergy between these two genera—and that its image is the homogeneous coordinate ring:

$$\mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2] / ((\mathbf{CaP}^2) \cdot (\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2))$$

of the union of the curve of Ochanine elliptic genera and the surface of Witten genera meeting with multiplicity 2 at the point $\mathbf{CaP}^2 = \mathbf{HP}^3 = \mathbf{HP}^2 = 0$ corresponding to the \hat{A} -genus. This all remains true if the word “oriented” is replaced with the word “spin” (Theorem 3). This paper also characterizes the Witten genus (tensor \mathbf{Q}) as the universal genus vanishing on total spaces of Cayley plane bundles (Theorem 1, a result proved independently by Dessai in [Des09].)

1. Introduction

This paper is inspired by two theorems. The first was proved in the 1950’s.

THEOREM (Chern-Hirzebruch-Serre [CHS57], Borel-Hirzebruch [BH59]). *The universal multiplicative genus for oriented manifolds is the signature:*

$$\text{MSO}_* \otimes \mathbf{Q} \rightarrow \text{MSO}_* / (E - F \cdot B) \otimes \mathbf{Q} \cong \mathbf{Q}[\sigma]$$

where $\deg(\sigma) = 4$.

Here and throughout this paper $(E - F \cdot B)$ denotes the \mathbf{Q} -vector space spanned by differences $E - F \cdot B$ where $F \rightarrow E \rightarrow B$ ranges over all fiber bundles with compact connected structure group. This vector space is in fact an ideal. The ambient bordism ring and hence the nature of the manifolds F, E, B will vary.

This theorem encapsulates several results. First of all Hirzebruch [Hir56] established a correspondence between genera $\text{MSO}_* \rightarrow \mathbf{Q}$ and formal power series $Q(z) = 1 + a_2 z^2 + a_4 z^4 + \cdots \in \mathbf{Q}[[z^2]]$. He then showed, in his celebrated Signature Theorem, that the signature is the genus corresponding to the power series:

$$Q(z) = z / \tanh(z) = 1 + \frac{1}{3} z^2 - \frac{1}{45} z^4 + \cdots$$

Next Chern-Hirzebruch-Serre [CHS57] proved that the signature is multiplicative, that is $\sigma(E) = \sigma(B) \sigma(F)$, for oriented fiber bundles with connected

structure group (or more generally with $\pi_1(B)$ acting trivially on $H^*(F, \mathbf{R})$). Finally Borel-Hirzebruch [BH59, Theorem 28.4] showed that the signature is the only multiplicative genus for oriented fiber bundles. Note that the Euler characteristic is multiplicative for oriented fiber bundles but is not an oriented bordism invariant (it is a complex bordism invariant). Totaro [Tot07] articulated the theorem as written above.

The second theorem which inspired this paper was proved in the 1980's.

THEOREM (Ochanine [Och87], Bott-Taubes [BT89]). *The universal multiplicative genus for spin manifolds is the Ochanine elliptic genus:*

$$\phi_{ell} : \mathrm{MSpin}_* \otimes \mathbf{Q} \rightarrow \mathrm{MSpin}_*/(E - F \cdot B) \otimes \mathbf{Q} \cong \mathbf{Q}[\delta, \varepsilon]$$

which maps onto the ring of modular forms on the congruence subgroup $\Gamma_0(2)$. In particular $\deg(\delta) = 2, \deg(\varepsilon) = 4$.

More concretely there is a family of \mathbf{Q} -valued multiplicative genera for spin manifolds and the members of this family correspond to stable elliptic curves with a marked point of order 2, the points of the weighted projective moduli space $\mathrm{Proj} \mathbf{Q}[\delta, \varepsilon]$. In particular the logarithms $g(y) = (y/Q(y))^{-1}$ of these genera are elliptic integrals:

$$\int_0^y \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}$$

The special cases $[\delta, \varepsilon] = [1, 1]$ and $[-\frac{1}{8}, 0]$ are the signature and \hat{A} genus respectively. The discriminant $\Delta = 64\varepsilon^2(\delta^2 - \varepsilon)$ vanishes in these cases so they correspond to singular elliptic curves.

A more elegant approach is to gather the entire family into a single characteristic power series whose coefficients are modular forms:

$$Q_{ell}(z) = \exp \left(\sum_{k=1}^{\infty} \frac{2}{(2k)!} \tilde{\mathbf{G}}_{2k} z^{2k} \right)$$

where $\tilde{\mathbf{G}}_k$ denotes the Eisenstein series:

$$\tilde{\mathbf{G}}_k = -\frac{1}{2k} \mathbf{B}_k + \sum_{n \geq 1} \left(\sum_{d|n} (-1)^{n/d} d^{k-1} \right) q^n$$

of weight k on the congruence subgroup $\Gamma_0(2)$ (see [Zag88]).

Inspired by this characteristic power series, Witten [Wit88] introduced the characteristic power series:

$$Q_W(z) = \exp \left(\sum_{k=1}^{\infty} \frac{2}{(2k)!} \mathbf{G}_{2k} z^{2k} \right)$$

where \mathbf{G}_k denotes the Eisenstein series:

$$\mathbf{G}_k = -\frac{1}{2k} \mathbf{B}_k + \sum_{n \geq 1} \left(\sum_{d|n} d^{k-1} \right) q^n = \frac{(k-1)!}{(2\pi i)^k} \mathbf{G}_k = \frac{\zeta(k)(k-1)!}{(2\pi i)^k} \mathbf{E}_k$$

of weight k on the full modular group $\mathrm{PSL}(2, \mathbf{Z})$. This defines the *Witten genus*:

$$\phi_W : \mathrm{MSO}_* \otimes \mathbf{Q} \rightarrow \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$$

which maps onto the ring of quasi-modular forms (\mathbf{G}_2 is not modular).

Now the bordism rings MSO_* and MSpin_* are the second and third terms in an infinite sequence:

$$\text{MO}_* \quad \text{MO}\langle 2 \rangle_* \quad \text{MO}\langle 4 \rangle_* \quad \text{MO}\langle 8 \rangle_* \quad \text{MO}\langle 9 \rangle_* \quad \cdots$$

Here $\text{MO}\langle n \rangle_*$ denotes the bordism ring of $\text{O}\langle n \rangle$ manifolds. An $\text{O}\langle n \rangle$ manifold is a smooth manifold M equipped with a lift of its stable tangent bundle's classifying map to the $(n-1)$ -connected cover $\text{BO}\langle n \rangle$ of the classifying space BO :

$$\begin{array}{ccc} & & \text{BO}\langle n \rangle \\ & \nearrow & \downarrow \\ M & \longrightarrow & \text{BO} \end{array}$$

The integers appearing in the sequence come from Bott periodicity:

$$\pi_i(\text{BO}) = \begin{cases} \mathbf{Z}/2 & \text{for } i = 1, 2 \bmod 8 \\ \mathbf{Z} & \text{for } i = 4, 8 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

The fourth term in the sequence $\text{MO}\langle 8 \rangle_*$ is sometimes denoted MString_* . An $\text{O}\langle 8 \rangle$ manifold can be characterized as a spin manifold whose characteristic class $\frac{1}{2}p_1$, the pullback of the generator of $H^4(\text{BO}\langle 8 \rangle, \mathbf{Z})$, equals zero.

In light of this sequence of bordism rings the two theorems above suggest the following question.

QUESTION. *What is the universal multiplicative genus for $\text{O}\langle 8 \rangle$ manifolds?*

$$\text{MO}\langle 8 \rangle_* \otimes \mathbf{Q} \rightarrow \text{MO}\langle 8 \rangle_*/(E - F \cdot B) \otimes \mathbf{Q}$$

This paper gives a first approximation to the answer. The idea came while reading Hirzebruch's textbook [HBJ92]. In §4.6 he shows that although the natural habitat of the elliptic genus is MSpin_* , it can already be observed in MSO_* . The result is originally due to Ochanine [Och87].

THEOREM (Ochanine).

$$\begin{aligned} \text{MSO}_*/(E - \mathbf{CP}^2 \cdot B) \otimes \mathbf{Q} &\cong \mathbf{Q}[\sigma] \\ \text{MSO}_*/(E - \mathbf{CP}^3 \cdot B) \otimes \mathbf{Q} &\cong \mathbf{Q}[\delta, \varepsilon] \end{aligned}$$

The point is that \mathbf{CP}^2 and \mathbf{CP}^3 both have lots of automorphisms and therefore are fibers of lots of bundles. But \mathbf{CP}^3 is spin whereas \mathbf{CP}^2 is not.

This made me wonder what would happen if I replaced \mathbf{CP}^3 with some $\text{O}\langle 8 \rangle$ manifold having lots of automorphisms. The Cayley plane $\mathbf{CaP}^2 = \text{F}_4/\text{Spin}(9)$ is such a manifold. Sometimes denoted \mathbf{OP}^2 it is in a certain sense a projective plane over the octonions (see [CS03, §12.2]). So I set out to compute the quotient:

$$\text{MSO}_*/(E - \mathbf{CaP}^2 \cdot B) \otimes \mathbf{Q}$$

I began by doing explicit power series calculations in the spirit of Hirzebruch's textbook [HBJ92] to determine all strictly multiplicative genera for Cayley plane bundles. My calculations strongly suggested that there were two families of strictly multiplicative genera, and I recognized them as the elliptic genus and the Witten genus.

I was not surprised to find the elliptic genus and the Witten genus. As the second theorem above states, the elliptic genus is known to be multiplicative not only for Cayley plane bundles but for *any* oriented fiber bundle with fiber a spin manifold and compact connected structure group. This is a consequence of its rigidity [BT89]. The Witten genus is also rigid but it is multiplicative in an even starker sense: the Witten genus of any Cayley plane bundle, and more generally any bundle with fiber a homogeneous space which is $O\langle 8 \rangle$, is zero (see [Sto96, Theorem 3.1]).

I *was* surprised, however, to find *only* the elliptic genus and the Witten genus. This clue led me to the following two theorems.

THEOREM 1. *The Witten genus is the universal genus vanishing on Cayley plane bundles. In other words the Witten genus is the quotient map:*

$$\phi_W : \text{MSO}_* \otimes \mathbf{Q} \rightarrow \text{MSO}_*/(E) \otimes \mathbf{Q} \cong \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$$

where $(E) \subset \text{MSO}_* \otimes \mathbf{Q}$ denotes the \mathbf{Q} -vector space spanned by total spaces of Cayley plane bundles $\mathbf{CaP}^2 \rightarrow E \rightarrow B$ with compact connected structure group. (This vector space is an ideal.)

THEOREM 2. *The universal multiplicative genus for Cayley plane bundles is the product $\phi_{\text{ell}} \times \phi_W$ of the Ochanine elliptic genus and the Witten genus. More precisely, the quotient $\text{MSO}_*/(E - \mathbf{CaP}^2 \cdot B) \otimes \mathbf{Q}$ injects into $\mathbf{Q}[\delta, \varepsilon] \times \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$ and the composition:*

$$\text{MSO}_* \otimes \mathbf{Q} \rightarrow \text{MSO}_*/(E - \mathbf{CaP}^2 \cdot B) \otimes \mathbf{Q} \hookrightarrow \mathbf{Q}[\delta, \varepsilon] \times \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$$

is $\phi_{\text{ell}} \times \phi_W$. Its image can be described geometrically as the weighted homogeneous coordinate ring:

$$\mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2] / ((\mathbf{CaP}^2) \cdot (\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2))$$

of the union of the weighted projective spaces:

$$\text{Proj } \mathbf{Q}[\delta, \varepsilon] \xleftarrow[\cong]{\phi_{\text{ell}}} \text{Proj } \mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2] / (\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2)$$

$$\text{Proj } \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6] \xleftarrow[\cong]{\phi_W} \text{Proj } \mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2] / (\mathbf{CaP}^2)$$

The first is the curve of elliptic genera (the moduli space of stable elliptic curves with a marked 2-division point). The second is the surface of Witten genera (related to the moduli space of stable elliptic curves). They meet with multiplicity 2 at the point $\mathbf{CaP}^2 = \mathbf{HP}^3 = \mathbf{HP}^2 = 0$ corresponding to the \hat{A} genus.

I should emphasize that $\phi_{\text{ell}} \times \phi_W$ is *not* surjective. Indeed an essential point is that:

$$\phi_{\text{ell}} \times \phi_W(\mathbf{CaP}^2) = (\varepsilon^2, 0)$$

but that $(\varepsilon, 0)$ is not in the image of $\phi_{\text{ell}} \times \phi_W$. For instance:

$$\phi_{\text{ell}} \times \phi_W(\mathbf{HP}^2) = (\varepsilon, 2\mathbf{G}_2^2 - \frac{5}{6}\mathbf{G}_4)$$

Thus there is a synergy between the elliptic genus and the Witten genus: individually they cannot recognize \mathbf{CaP}^2 as an indecomposable but together they can.

Note that the values:

$$\phi_W(\mathbf{CaP}^2) = 0 \quad \phi_{ell}(\mathbf{HP}^3) = 0 \quad \phi_{ell}(\mathbf{CaP}^2) = \varepsilon^2 = \phi_{ell}(\mathbf{HP}^2)^2$$

together with:

$$\phi_{ell} \times \phi_W(\text{Kum}) = (16\delta, 48\mathbf{G}_2) \quad \phi_{ell} \times \phi_W(\mathbf{HP}^3) = (0, -\frac{4}{9}\mathbf{G}_2^3 + \frac{1}{9}\mathbf{G}_2\mathbf{G}_4 + \frac{7}{1080}\mathbf{G}_6)$$

account for the isomorphisms of weighted projective spaces asserted in the theorem (compare Proposition 9).

There is further evidence that Theorems 1 & 2 are a good approximation of the answer to the QUESTION above.

THEOREM 3. *Theorems 1 & 2 remain true if MSO_* is replaced with MSpin_* .*

Note that the QUESTION would be answered if MSpin_* could be replaced with $\text{MO}\langle 8 \rangle_*$. (The description of the image in Theorem 2 would need to be modified though.)

Note also that Dessai proved Theorem 1 independently in [Des09]. In fact he showed that it remains true if MSO_* is replaced with $\text{MO}\langle 8 \rangle_*$ (in which case the generator \mathbf{G}_2 should be erased). Note that Dessai asked (Problem 4.2 of his paper) for a geometric description of the universal multiplicative genus for \mathbf{CaP}^2 bundles. Theorem 2 answers that question as stated. However, I expect a richer answer to come from replacing MSpin with $\text{MO}\langle 8 \rangle$ in Theorem 2.

I conclude the introduction by speculating about how these results might be relevant to homotopy theory. Kreck-Stolz [KS93] computed:

$$\text{MSpin}_*/(E) \cong \text{KO}_*(\text{pt}) \quad \text{MSpin}_*/(E - \mathbf{HP}^2 \cdot B) \otimes \mathbf{Z}[\frac{1}{2}] \cong \mathbf{Z}[\frac{1}{2}][\delta, \varepsilon]$$

where in both cases $\mathbf{HP}^2 \rightarrow E \rightarrow B$ ranges over all bundles with compact connected structure group. They used these calculations to give alternate constructions of KO -theory and elliptic homology (and in so doing defined elliptic cohomology with \mathbf{Z} rather than $\mathbf{Z}[\frac{1}{2}]$ coefficients, which was novel). They suggested (see [KS93, p. 235]) that replacing MSpin_* and \mathbf{HP}^2 with $\text{MO}\langle 8 \rangle_*$ and \mathbf{CaP}^2 in their constructions might result in homology theories as well. (Sati's recent paper [Sat09] explores the relevance of such a theory to string theory.) Theorems 1 & 2 suggest that the first might be closely related to topological modular forms [Hop02] while the second might be some sort of hybrid of elliptic homology and topological modular forms.

2. Cayley plane bundles

Before we can prove Theorems 1 & 2, we need to discuss Cayley plane bundles in general. The Cayley plane is the homogeneous space $\mathbf{CaP}^2 = \text{F}_4/\text{Spin}(9)$. Much of what follows applies to any bundle with fiber a homogeneous space G/H though so we begin in that generality and later specialize to the case $G/H = \text{F}_4/\text{Spin}(9)$.

Throughout this section let G be a compact connected Lie group, let $i_{H,G} : H \hookrightarrow G$ be a maximal rank subgroup, and let $i_{T,H} : T \rightarrow H$ and $i_{T,G} : T \rightarrow G$ be the inclusions of a common maximal torus.

Every G/H bundle (with structure group G) pulls back from the universal G/H bundle $G/H \rightarrow BH \rightarrow BG$. That is, every G/H bundle fits into a pullback diagram:

$$\begin{array}{ccc} E_f & \xrightarrow{g} & BH \\ \pi_f \downarrow & & \downarrow \text{Bi}_{H,G} \\ Z & \xrightarrow{f} & BG \end{array}$$

where f is unique up to homotopy and g is canonically determined by f .

Let η denote the relative tangent bundle of $BH \rightarrow BG$. Then the relative tangent bundle of $E_f \rightarrow Z$ is the pullback $g^*(\eta)$ and there is an exact sequence:

$$0 \rightarrow g^*(\eta) \rightarrow TE_f \rightarrow \pi_f^* TZ \rightarrow 0$$

This implies for instance that $p_1(TE_f) = \pi_f^* p_1(TZ) + g^* p_1(\eta)$.

The characteristic classes of η , or rather their pullbacks to $H^*(BT, \mathbf{Z})$, can be computed using the beautiful methods of [BH58] (see especially Theorem 10.7). For instance the pullbacks of the first Pontrjagin class $p_1(\eta)$ and more generally the Pontrjagin class $s_I(p)(\eta)$ can be computed using the formulas:

$$\text{Bi}_{T,H}^* p_1(\eta) = \sum r_i^2 \quad \text{Bi}_{T,H}^* s_I(p)(\eta) = s_I(r_1^2, \dots, r_m^2)$$

where $(\pm r_1, \dots, \pm r_m)$ are the roots of G complementary to those of H regarded as elements of $H^*(BT, \mathbf{Z})$.

Borel-Hirzebruch's Lie-theoretic description [BH58, BH59] of the pushforward:

$$\text{Bi}_{H,G*} : H^*(BH, \mathbf{Z}) \rightarrow H^*(BG, \mathbf{Z})$$

is essential to proving Theorems 1 & 2. In order to state their result we need to introduce some notation.

Associated to G is a generalized Euler class $\tilde{e}(G/T) \in H^*(BT, \mathbf{Z})$. It makes sense to call it that because it restricts to the Euler class of the fiber G/T of the bundle $BT \rightarrow BG$. Up to sign $\tilde{e}(G/T)$ is the product of a set of positive roots of G , regarded as elements of $H^*(BT, \mathbf{Z})$. More precisely it is the product of the roots of an invariant almost complex structure on G/T . (See [BH58, §12.3, §13.4] for more details.) Note that G/T always admits a complex structure and that although the individual roots associated to an almost complex structure depend on the almost complex structure, their product $\tilde{e}(G/T)$ does not.

THEOREM 4 (Borel-Hirzebruch, Theorem 20.3 of [BH59]). *If $t \in H^*(BT, \mathbf{Z})$ then:*

$$\text{Bi}_{T,G}^* \text{Bi}_{T,G*}(t) = \frac{1}{\tilde{e}(G/T)} \sum_{w \in W(G)} \text{sgn}(w) w(t)$$

COROLLARY 5. *If $h \in H^*(BH, \mathbf{Z})$ then:*

$$\text{Bi}_{T,G}^* \text{Bi}_{H,G*}(h) = \sum_{[w] \in W(G)/W(H)} w \left(\frac{\tilde{e}(H/T)}{\tilde{e}(G/T)} \text{Bi}_{T,H}^*(h) \right)$$

where the sum runs over the cosets of $W(H)$ in $W(G)$.

PROOF. Since $\text{Bi}_{T,H*}\tilde{e}(H/T) = \chi(H/T) = |W(H)| \in H^0(BH, \mathbf{Z})$, write:

$$\text{Bi}_{T,G}^* \text{Bi}_{H,G*}(h) = \text{Bi}_{T,G}^* \text{Bi}_{H,G*} \left(\frac{\text{Bi}_{T,H*}(\tilde{e}(H/T))}{|W(H)|} \cdot h \right)$$

Apply the projection formula:

$$\begin{aligned} &= \frac{1}{|W(H)|} \text{Bi}_{T,G}^* \text{Bi}_{H,G*} \text{Bi}_{T,H*}(\tilde{e}(H/T) \cdot \text{Bi}_{T,H}^*(h)) \\ &= \frac{1}{|W(H)|} \text{Bi}_{T,G}^* \text{Bi}_{T,G*}(\tilde{e}(H/T) \cdot \text{Bi}_{T,H}^*(h)) \end{aligned}$$

Apply Theorem 4:

$$= \frac{1}{|W(H)|} \cdot \frac{1}{|\tilde{e}(G/T)|} \sum_{w \in W(G)} \text{sgn}(w) w(\tilde{e}(H/T) \cdot \text{Bi}_{T,H}^*(h))$$

Since $w(\tilde{e}(G/T)) = \text{sgn}(w)\tilde{e}(G/T)$:

$$= \frac{1}{|W(H)|} \sum_{w \in W(G)} w \left(\frac{\tilde{e}(H/T)}{\tilde{e}(G/T)} \text{Bi}_{T,H}^*(h) \right)$$

Since $W(G)$ acts on $H^*(BT, \mathbf{Z})$ by ring homomorphisms, since if $w \in W(H)$ then $w(\tilde{e}(H/T)) = \text{sgn}(w)\tilde{e}(H/T)$ and $w(\tilde{e}(G/T)) = \text{sgn}(w)\tilde{e}(G/T)$, and since $\text{Bi}_{T,H}^*$ maps to the $W(H)$ -invariant subring of $H^*(BT, \mathbf{Z})$, this sum can be written over the cosets of $W(H)$ in $W(G)$:

$$= \sum_{[w] \in W(G)/W(H)} w \left(\frac{\tilde{e}(H/T)}{\tilde{e}(G/T)} \text{Bi}_{T,H}^*(h) \right) \quad \square$$

Now we specialize to Cayley plane bundles. Let F_4 denote the 1-connected compact Lie group of type F_4 . The extended Dynkin diagram of F_4 is:

$$\begin{array}{ccccccccc} \bullet & \text{---} & \circ & \text{---} & \circ & \Rightarrow & \circ & \text{---} & \circ \\ -\tilde{a} & & a_1 & & a_2 & & a_3 & & a_4 \end{array}$$

The corresponding simple roots can be taken to be:

$$a_1 = e_2 - e_3 \quad a_2 = e_3 - e_4 \quad a_3 = e_4 \quad a_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$$

Since the coefficient of a_4 in the maximal root $\tilde{a} = 2a_1 + 3a_2 + 4a_3 + 2a_4 = e_1 + e_2$ is prime, a theorem of Borel & de Siebenthal [BDS49] implies that erasing a_4 from the extended Dynkin diagram gives the Dynkin diagram of a subgroup:

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \circ & \Rightarrow & \circ \\ -\tilde{a} & & a_1 & & a_2 & & a_3 \end{array}$$

Since F_4 is 1-connected this subgroup is $\text{Spin}(9)$, the 1-connected double cover of $\text{SO}(9)$. The Cayley plane is the homogeneous space $\mathbf{CaP}^2 = F_4/\text{Spin}(9)$.

In terms of the standard basis e_1, \dots, e_4 , the roots of $\text{Spin}(9)$ are:

$$\begin{cases} \pm e_i & 1 \leq i \leq 4 \\ \pm e_i \pm e_j & 1 \leq i < j \leq 4 \end{cases}$$

The roots of F_4 are those of $\text{Spin}(9)$ together with the complementary roots:

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$$

The following positive roots define an almost complex structure on $\text{Spin}(9)/T$:

$$\begin{cases} e_i & 1 \leq i \leq 4 \\ e_i \pm e_j & 1 \leq i < j \leq 4 \end{cases}$$

These positive roots together with the following complementary positive roots define an almost complex structure on F_4/T :

$$r_i := \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \quad \text{for } 1 \leq i \leq 8$$

In order to identify these roots with elements of $H^2(BT, \mathbf{Z}) \cong \text{Hom}(\Gamma, \mathbf{Z})$ note that in general a Lie group's lattice of integral forms is sandwiched somewhere between its root and weight lattices:

$$R \subset \text{Hom}(\Gamma, \mathbf{Z}) \subset W \subset LT^*$$

But in the case of F_4 all three lattices coincide (because the Cartan matrix of F_4 has determinant 1).

Finally note that if s_i denotes reflection across the hyperplane orthogonal to the simple root a_i then the 3 cosets of $W(\text{Spin}(9))$ in $W(F_4)$ can be represented by the reflections $\{1, s_4, s_4 s_3 s_4\}$ which act on e_1, \dots, e_4 according to the matrices:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \right\}$$

In particular these reflections act on the set of positive complementary roots r_i by:

$$\begin{aligned} \{r_i\} &= \left\{ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\} \\ s_4(\{r_i\}) &= \left\{ e_1, e_2, e_3, e_4, \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \frac{1}{2}(e_1 + e_2 - e_3 + e_4), \right. \\ &\quad \left. \frac{1}{2}(e_1 - e_2 + e_3 + e_4), \frac{1}{2}(-e_1 + e_2 + e_3 + e_4) \right\} \\ s_4 s_3 s_4(\{r_i\}) &= \left\{ e_1, e_2, e_3, e_4, \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \right. \\ &\quad \left. \frac{1}{2}(e_1 - e_2 + e_3 - e_4), \frac{1}{2}(-e_1 + e_2 + e_3 - e_4) \right\} \end{aligned}$$

COROLLARY 6.

$$Bi_{T, F_4}^* Bi_{\text{Spin}(9), F_4} s_I(p)(\eta) = \frac{s_I(r_1^2, \dots, r_8^2)}{\prod_i r_i} + s_4 \left(\frac{s_I(r_1^2, \dots, r_8^2)}{\prod_i r_i} \right) + s_4 s_3 s_4 \left(\frac{s_I(r_1^2, \dots, r_8^2)}{\prod_i r_i} \right)$$

where the complementary roots $r_i = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$ are regarded as elements of $H^2(BT, \mathbf{Z})$ and $s_4, s_4 s_3 s_4$ act on them as described above.

3. Proof of Theorem 1

Theorem 1 is a consequence of Proposition 7 together with the calculation (Proposition 9 in the next section) of the Witten genus of $\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3$.

PROPOSITION 7. *If $n \geq 4$ then there is a Cayley plane bundle $\mathbf{CaP}^2 \rightarrow E_n \rightarrow \mathbf{HP}^{n-4}$ with $s_n(p)[E_n] \neq 0$.*

The proof relies on the following lemma (Lemma 16.2 of [MS74]).

LEMMA 8 (Thom). *If $0 \rightarrow V_1 \rightarrow W \rightarrow V_2 \rightarrow 0$ is an exact sequence of vector bundles then:*

$$s_I(p)(W) = \sum_{JK=I} s_J(p)(V_1) s_K(p)(V_2)$$

where the sum ranges over all partitions J and K with juxtaposition JK equal to I .

PROOF OF PROPOSITION 7. Recall that the extended Dynkin diagram of F_4 is:

$$\begin{array}{ccccccc} \bullet & \text{---} & \circ & \text{---} & \circ & \Rightarrow\Rightarrow & \circ & \text{---} & \circ \\ -\tilde{a} & & a_1 & & a_2 & & a_3 & & a_4 \end{array}$$

Since the coefficient of a_1 in the maximal root $\tilde{a} = 2a_1 + 3a_2 + 4a_3 + 2a_4$ is prime, a theorem of Borel-Siebenthal [BDS49] implies that erasing a_1 from the extended Dynkin diagram gives the Dynkin diagram of a subgroup. This subgroup's (half) extended Dynkin diagram is:

$$\begin{array}{ccccccc} \circ & & \circ & \Rightarrow\Rightarrow & \circ & \text{---} & \circ & \Leftarrow\Leftarrow & \bullet \\ -\tilde{a} & & a_2 & & a_3 & & a_4 & & -\tilde{b} \end{array}$$

Since the coefficient of a_3 in the maximal root $\tilde{b} = 2a_4 + 2a_3 + a_2$ is prime, the same theorem implies that F_4 has a subgroup with Dynkin diagram:

$$\begin{array}{ccccccc} \circ & & \circ & & \circ & \Leftarrow\Leftarrow & \circ \\ -\tilde{a} & & a_2 & & a_4 & & -\tilde{b} \end{array}$$

Passing to this subgroup's 1-connected cover gives a map:

$$h : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(2) \rightarrow F_4$$

This map h restricts to a double covering $h|_T$ of compatible maximal tori whose induced homomorphism on H^1 corresponds to the inclusion of weight lattices:

$$\mathbf{Z}\langle a_1, a_2, a_3, a_4 \rangle \hookrightarrow \mathbf{Z}\langle -\frac{1}{2}\tilde{a}, \frac{1}{2}a_2, a_4 - \frac{1}{2}\tilde{b}, a_4 - \tilde{b} \rangle = \mathbf{Z}\langle a_1, \frac{1}{2}a_2, a_3, a_4 \rangle$$

Let $f : \mathbf{HP}^{n-4} \rightarrow \mathrm{BF}_4$ denote the composition:

$$\mathbf{HP}^{n-4} \hookrightarrow \mathbf{HP}^\infty = \mathrm{BSp}(1) \xrightarrow{\mathrm{Bi}_1} \mathrm{B}(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(3)) \xrightarrow{\mathrm{B}h} \mathrm{BF}_4$$

where $i_1 : \mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(3)$ is the inclusion of the first factor. The map f classifies a Cayley plane bundle $\mathbf{CaP}^2 \rightarrow E_n \rightarrow \mathbf{HP}^{n-4}$ fitting into a pull-back diagram:

$$\begin{array}{ccc} E_n & \xrightarrow{g} & \mathrm{BSpin}(9) \\ \pi_f \downarrow & & \downarrow \mathrm{Bi}_{\mathrm{Spin}(9), F_4} \\ \mathbf{HP}^{n-4} & \xrightarrow{f} & \mathrm{BF}_4 \end{array}$$

Use this diagram to compute:

$$\begin{aligned} s_n(p)[E_n] &= \int_{E_n} s_n(p)(TE_n) = \int_{E_n} s_n(p)(\pi_f^* T\mathbf{HP}^{n-4} \oplus g^*(\eta)) = \int_{E_n} g^* s_n(p)(\eta) \quad (\text{Lemma 8}) \\ &= \int_{\mathbf{HP}^{n-4}} \pi_{f*} g^* s_n(p)(\eta) = \int_{\mathbf{HP}^{n-4}} f^* \text{Bi}_{\text{Spin}(9), \mathbf{F}_4} s_n(p)(\eta) \end{aligned}$$

Since the inclusion of the maximal torus $i_{S^1, \text{Sp}(1)} : S^1 \hookrightarrow \text{Sp}(1)$ induces an injection:

$$H^*(\text{BSp}(1), \mathbf{Z}) = \mathbf{Z}[\frac{1}{4}\tilde{a}^2] \hookrightarrow \mathbf{Z}[\frac{1}{2}\tilde{a}] = H^*(\text{BS}^1, \mathbf{Z})$$

the pullback of $\text{Bi}_{\text{Spin}(9), \mathbf{F}_4} s_n(p)(\eta) \in H^{4n}(\text{BF}_4, \mathbf{Z})$ to $H^{4n}(\text{BSp}(1), \mathbf{Z})$ can be computed by pulling along the bottom of the diagram:

$$\begin{array}{ccccc} \text{BSp}(1) & \xrightarrow{\text{Bi}_1} & \text{B}(\text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(2)) & \xrightarrow{\text{B}h} & \text{BF}_4 \\ \uparrow & & \uparrow & & \uparrow \\ \text{BS}^1 & \xrightarrow{\text{B}(i_1|_{S^1})} & \text{BT}^4 & \xrightarrow{\text{B}(h|_T)} & \text{BT}^4 \end{array}$$

Recall that Corollary 6 gives a formula for the image of $\text{Bi}_{\text{Spin}(9), \mathbf{F}_4} s_n(p)(\eta)$ in $H^{4n}(\text{BT}^4, \mathbf{Z})$. The composition $\text{B}(i_1|_{S^1})^* \circ \text{B}(h|_T)^*$ extracts the coefficient of $\frac{1}{2}\tilde{a}$ with respect to the basis $\{\tilde{a}, a_2, a_4, \tilde{b}\}$. Since:

$$(e_1, e_2, e_3, e_4) = \left(\frac{1}{2}(\tilde{a} + \tilde{b}), \frac{1}{2}(\tilde{a} - \tilde{b}), \frac{1}{2}(a_2 - 2a_4 - \tilde{b}), -\frac{1}{2}(a_2 + 2a_4 + \tilde{b}) \right)$$

it follows that the integral $\int_{\mathbf{HP}^{n-4}} f^* \text{Bi}_{\text{Spin}(9), \mathbf{F}_4} s_n(p)(\eta)$ can be computed by taking the formula of Corollary 6 and substituting $(e_1, e_2, e_3, e_4) \mapsto (1, 1, 0, 0)$. Some care is needed though since these substitutions make denominators vanish. Substituting $(e_1, e_2, e_3, e_4) \mapsto (1, 1 + z, z, z)$ and applying l'Hôpital's rule six times with respect to z gives:

$$s_n(p)[E_n] = -\frac{1}{3}(n-3)n(2n-1)(2n+1)$$

which is strictly negative for $n \geq 4$. \square

4. Proof of Theorem 2

As explained in the introduction, the elliptic genus and the Witten genus:

$$\begin{aligned} \phi_{ell} : \text{MSO}_* \otimes \mathbf{Q} &\rightarrow \mathbf{Q}[\delta, \varepsilon] \\ \phi_W : \text{MSO}_* \otimes \mathbf{Q} &\rightarrow \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6] \end{aligned}$$

are both known to be multiplicative for Cayley plane bundles. This implies that the ideal $I = (E - \text{CaP}^2 \cdot B)$ is contained in the kernel K of the product:

$$\phi_{ell} \times \phi_W : \text{MSO}_* \otimes \mathbf{Q} \rightarrow \mathbf{Q}[\delta, \varepsilon] \times \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$$

To prove Theorem 2, we compute K and then show that $I = K$.

The inclusion:

$$\mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \text{CaP}^2] \rightarrow \text{MSO}_* \otimes \mathbf{Q}$$

is an isomorphism in degrees 0 through 16. (The same is true if MSO_* is replaced with MSpin_* .) Here Kum denotes the Kummer surface, a 4-dimensional spin manifold with signature 16.

PROPOSITION 9.

$$\begin{aligned}\phi_{ell} \times \phi_W(\text{Kum}) &= (16\delta, 48\mathbf{G}_2) \\ \phi_{ell} \times \phi_W(\mathbf{HP}^2) &= (\varepsilon, 2\mathbf{G}_2^2 - \frac{5}{6}\mathbf{G}_4) \\ \phi_{ell} \times \phi_W(\mathbf{HP}^3) &= (0, -\frac{4}{9}\mathbf{G}_2^3 + \frac{1}{9}\mathbf{G}_2\mathbf{G}_4 + \frac{7}{1080}\mathbf{G}_6) \\ \phi_{ell} \times \phi_W(\mathbf{CaP}^2) &= (\varepsilon^2, 0)\end{aligned}$$

PROOF. Use the following identities in $\text{MSO}_* \otimes \mathbf{Q}$:

$$\begin{aligned}\text{Kum} &= 16\mathbf{CP}^2 \\ \mathbf{HP}^2 &= 3(\mathbf{CP}^2)^2 - 2\mathbf{CP}^4 \\ \mathbf{HP}^3 &= \frac{2}{3}(\mathbf{CP}^2)^3 - \mathbf{CP}^2\mathbf{CP}^4 + \frac{1}{3}\mathbf{CP}^6 \\ \mathbf{CaP}^2 &= \frac{145}{3}(\mathbf{CP}^2)^4 - 92(\mathbf{CP}^2)^2 \cdot \mathbf{CP}^4 + 36\mathbf{CP}^2 \cdot \mathbf{CP}^6 + 18(\mathbf{CP}^4)^2 - \frac{28}{3}\mathbf{CP}^8\end{aligned}$$

To verify the first identity note that the signature restricts to an isomorphism $\text{MSO}_4 \rightarrow \mathbf{Z}$. To verify the rest, compare Pontrjagin numbers. For the Pontrjagin numbers of \mathbf{CP}^n use the formula $p(T\mathbf{CP}^n) = (1 - g^2)^{n+1}$ where g generates $H^2(\mathbf{CP}^n, \mathbf{Z})$. For the Pontrjagin numbers of \mathbf{HP}^n use Hirzebruch's formula $p(T\mathbf{HP}^n) = (1 + u)^{2n+2}(1 + 4u)^{-1}$ where u generates $H^4(\mathbf{HP}^n, \mathbf{Z})$ (Theorem 1.3 of [HBJ92]). For the Pontrjagin numbers of \mathbf{CaP}^2 see [BH58, Theorem 19.4].

The values $\phi_{ell} \times \phi_W(\mathbf{CP}^{2n})$ can in turn be extracted from the characteristic power series of ϕ_{ell} and ϕ_W since any genus ϕ with characteristic power series Q satisfies:

$$g'(z) = \frac{d}{dz} \left(\frac{z}{Q(z)} \right)^{-1} = \sum_{n=1}^{\infty} \phi(\mathbf{CP}^{2n}) z^{2n}$$

where the logarithm $g(z) = (z/Q(z))^{-1}$ is the formal power series satisfying $g(z/Q(z)) = 1$.

To extract $\phi_{ell} \times \phi_W(\mathbf{CP}^{2n})$ for $1 \leq n \leq 4$ from the characteristic power series given in the introduction, use the identities:

$$\begin{aligned}\delta &= 3\tilde{\mathbf{G}}_2 & \tilde{\mathbf{G}}_6 &= \frac{120}{7}(4\tilde{\mathbf{G}}_2^3 - \tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_4) & \mathbf{G}_8 &= 120\mathbf{G}_4^2 \\ \varepsilon &= \frac{1}{6}(12\tilde{\mathbf{G}}_2^2 - 5\tilde{\mathbf{G}}_4) & \tilde{\mathbf{G}}_8 &= -\frac{20}{3}(144\tilde{\mathbf{G}}_2^4 - 120\tilde{\mathbf{G}}_2^2 + 7\tilde{\mathbf{G}}_4^2)\end{aligned} \quad \square$$

COROLLARY 10. *The kernel of the restriction:*

$$\phi_{ell} \times \phi_W : \mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2] \rightarrow \mathbf{Q}[\delta, \varepsilon] \times \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$$

is the ideal $(\mathbf{CaP}^2) \cdot (\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2)$.

PROOF. Proposition 9 implies that the restriction of ϕ_{ell} splits as a tensor product:

$$\mathbf{Q}[\text{Kum}] \otimes \mathbf{Q}[\mathbf{HP}^3] \otimes \mathbf{Q}[\mathbf{HP}^2, \mathbf{CaP}^2] \rightarrow \mathbf{Q}[\delta] \otimes \mathbf{Q} \otimes \mathbf{Q}[\varepsilon]$$

whose kernel is clearly the ideal $(\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2)$. It also implies that ϕ_W vanishes on \mathbf{CaP}^2 and restricts to an isomorphism $\mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3] \rightarrow \mathbf{Q}[\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$. The restriction of $\phi_{ell} \times \phi_W$ to $\mathbf{Q}[\text{Kum}, \mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2]$ therefore has kernel:

$$(\mathbf{CaP}^2) \cap (\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2) = (\mathbf{CaP}^2) \cdot (\mathbf{HP}^3, \mathbf{CaP}^2 - (\mathbf{HP}^2)^2) \quad \square$$

Corollary 10 implies that the kernel K of $\phi_{ell} \times \phi_W$ is generated by:

$$\begin{cases} R_7 = \mathbf{CaP}^2 \cdot \mathbf{HP}^3 \\ R_8 = \mathbf{CaP}^2 \cdot (\mathbf{CaP}^2 - (\mathbf{HP}^2)^2) \end{cases}$$

together with the differences $E_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-4}$ for $n \geq 4$ where $\mathbf{CaP}^2 \rightarrow E_n \rightarrow \mathbf{HP}^{n-4}$ is the bundle constructed in the proof of Theorem 1. The \mathbf{Q} -vector space:

$$V_n(K) = K_{4n} / \left(\sum_{0 < i < n} K_{4i} \cdot MSO_{4n-4i} \right)$$

therefore has dimension:

$$\dim_{\mathbf{Q}} V_n(K) = \begin{cases} 1 & \text{for } n \geq 9 \\ 2 & \text{for } 7 \leq n \leq 8 \\ 1 & \text{for } 5 \leq n \leq 6 \\ 0 & \text{for } 1 \leq n \leq 4 \end{cases}$$

To prove that $I = K$ it suffices to show that the \mathbf{Q} -vector space:

$$V_n(I) = I_{4n} / \left(\sum_{0 < i < n} I_{4i} \cdot MSO_{4n-4i} \right)$$

has the same dimension as $V_n(K)$ for each $n \geq 1$. Theorem 1 implies that $\dim_{\mathbf{Q}} V_n(I) \geq 1$ for $n \geq 5$ so all that remains is to show that $\dim_{\mathbf{Q}} V_n(I) = 2$ for $n = 7, 8$. We do this by constructing two bundles $\mathbf{CaP}^2 \rightarrow E'_7 \rightarrow \mathbf{HP}^2 \times \mathbf{HP}^1$, $\mathbf{CaP}^2 \rightarrow E'_8 \rightarrow \mathbf{HP}^3 \times \mathbf{HP}^1$ and showing that the images of $E_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-4}$ and $E'_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-5} \cdot \mathbf{HP}^1$ are linearly independent in $V_n(I)$ for $n = 7, 8$. To establish linear independence it suffices to exhibit two Pontrjagin numbers α_n, β_n which vanish on $\sum_{0 < i < n} I_{4i} \cdot MSO_{4n-4i}$ and to check that the determinant:

$$\begin{vmatrix} \alpha_n(E_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-4}) & \alpha_n(E'_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-5} \cdot \mathbf{HP}^1) \\ \beta_n(E_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-4}) & \beta_n(E'_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-5} \cdot \mathbf{HP}^1) \end{vmatrix}$$

is nonzero.

Modify the construction of $\mathbf{CaP}^2 \rightarrow E_n \rightarrow \mathbf{HP}^{n-4}$ in the proof of Theorem 1 as follows. Let $f : \mathbf{HP}^{n-5} \times \mathbf{HP}^1 \rightarrow \mathbf{BF}_4$ denote the composition:

$$\mathbf{HP}^{n-5} \times \mathbf{HP}^1 \hookrightarrow \mathbf{HP}^\infty \times \mathbf{HP}^\infty = \mathbf{B}(\mathbf{Sp}(1) \times \mathbf{Sp}(1)) \xrightarrow{\mathbf{B}(i_1 \times i_2)} \mathbf{B}(\mathbf{Sp}(1) \times \mathbf{Sp}(1) \times \mathbf{Sp}(3)) \xrightarrow{\mathbf{B}h} \mathbf{BF}_4$$

and let $\mathbf{CaP}^2 \rightarrow E'_n \rightarrow \mathbf{HP}^{n-5} \times \mathbf{HP}^1$ denote the Cayley plane bundle classified by f .

PROPOSITION 11.

$$\begin{aligned} s_7(p)[E'_7] &= -5824 & s_8(p)[E'_8] &= -15776 \\ s_{4,3}(p)[E'_7] &= 9184 & s_{4,4}(p)[E'_8] &= 11024 \end{aligned}$$

PROOF. The calculation of $s_n(p)[E_n]$ in the proof of Proposition 7 can be adapted to compute $s_n(p)[E'_n]$. Instead of substituting $(e_1, e_2, e_3, e_4) \mapsto (1, 1, 0, 0)$ into the formula of Corollary 6, substitute $(e_1, e_2, e_3, e_4) \mapsto (g_1, g_1, g_2, -g_2)$ where g_1, g_2 are indeterminants and then extract the coefficient of $g_1^{2n-10} g_2^2$. This coefficient can be extracted assuming $n \geq 6$ by differentiating twice with respect to

g_2 , dividing by 2, applying l'Hôpital's rule 6 times with respect to g_2 , and then substituting $(g_1, g_2) \mapsto (1, 0)$. This leads, for $n \geq 6$, to the formula:

$$s_n(p)[E'_n] = -\frac{1}{45}(n-4)n(2n-1)(2n+1)(2n^2-7n+15)$$

The numbers $s_{4,3}(p)[E'_7]$ and $s_{4,4}(p)[E'_8]$ can be computed similarly because, just as for $s_n(p)[E_n]$ and $s_n(p)[E'_n]$, the Pontrjagin class $p(\mathbf{HP}^{n-5} \times \mathbf{HP}^1)$ does not affect the calculation. For instance Lemma 8 implies that $s_3(p)(T(\mathbf{HP}^2 \times \mathbf{HP}^1)) = 0$ and hence that:

$$s_{4,3}(p)[E'_7] = \int_{E'_7} s_{4,3}(p)(TE'_7) = \int_{E'_7} s_{4,3}(p)(\pi_f^* T(\mathbf{HP}^2 \times \mathbf{HP}^1) \oplus g^*(\eta)) = \int_{E'_7} g^* s_{4,3}(p)(\eta)$$

The last integral can then be computed using the formula of Corollary 6 as above. \square

PROPOSITION 12.

$$s_{4,3}(p)[E_7] = 3164 + s_3[\mathbf{HP}^3] s_4[\mathbf{CaP}^2] \quad s_{4,4}(p)[E_8] = 2932 + s_4[\mathbf{HP}^4] s_4[\mathbf{CaP}^2]$$

PROOF. The calculation is similar to that of $s_7(p)[E_7]$ and $s_8(p)[E_8]$ except that the Pontrjagin numbers of the base space \mathbf{HP}^{n-4} begin to creep in. For instance:

$$\begin{aligned} s_{4,3}(p)[E_7] &= \int_{E_7} s_{4,3}(p)(TE_7) = \int_{E_7} s_{4,3}(p)(\pi_f^* T\mathbf{HP}^3 \oplus g^* \eta) \\ &= \int_{\mathbf{HP}^3} s_3(p)(T\mathbf{HP}^3) \cdot f^* \text{Bi}_{\text{Spin}(9), \mathbf{F}_4} s_4(p)(\eta) + f^* \text{Bi}_{\text{Spin}(9), \mathbf{F}_4} s_{4,3}(p)(\eta) \\ &= s_3[\mathbf{HP}^3] s_4[\mathbf{CaP}^2] + 3164 \end{aligned} \quad \square$$

In fact $s_4[\mathbf{CaP}^2] = -84$ and $s_n(p)[\mathbf{HP}^n] = -4^n + 2n + 2$ but these numbers drop out in the end.

PROPOSITION 13. If $(\alpha_7, \beta_7) = (s_7(p), s_{4,3}(p))$ and $(\alpha_8, \beta_8) = (s_8(p), s_{4,4}(p))$ then α_n and β_n vanish on $\sum_{0 < i < n} I_{4i} \cdot \text{MSO}_{4n-4i}$ and the determinant:

$$\begin{vmatrix} \alpha_n(E_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-4}) & \alpha_n(E'_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-5} \cdot \mathbf{HP}^1) \\ \beta_n(E_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-4}) & \beta_n(E'_n - \mathbf{CaP}^2 \cdot \mathbf{HP}^{n-5} \cdot \mathbf{HP}^1) \end{vmatrix}$$

is nonzero for $n = 7, 8$.

PROOF. If $n = 7$ then the \mathbf{Q} -vector space $\sum_{0 < i < n} I_{4i} \cdot \text{MSO}_{4n-4i}$ is spanned by bordism classes of the form:

$$\begin{aligned} (E_5 - \mathbf{HP}^1 \cdot \mathbf{CaP}^2) \cdot M_2 \\ (E_6 - \mathbf{HP}^2 \cdot \mathbf{CaP}^2) \cdot M_1 \end{aligned}$$

where M_i denotes a closed oriented manifold of real dimension $4i$. Lemma 8 implies that $\alpha_7 = s_7(p)$ and $\beta_7 = s_{4,3}(p)$ vanish on all such bordism classes. By Propositions 11 & 12:

$$\begin{vmatrix} \alpha_7(E_7 - \mathbf{CaP}^2 \cdot \mathbf{HP}^3) & \alpha_7(E'_7 - \mathbf{CaP}^2 \cdot \mathbf{HP}^2 \cdot \mathbf{HP}^1) \\ \beta_7(E_7 - \mathbf{CaP}^2 \cdot \mathbf{HP}^3) & \beta_7(E'_7 - \mathbf{CaP}^2 \cdot \mathbf{HP}^2 \cdot \mathbf{HP}^1) \end{vmatrix} = \begin{vmatrix} -1820 & -5824 \\ 3164 & 9184 \end{vmatrix} \neq 0$$

If $n = 8$ then the case $n = 7$ proved above implies that $\sum_{0 < i < n} I_{4i} \cdot \text{MSO}_{4n-4i}$ is spanned by bordism classes of the form:

$$\begin{aligned} & (E_5 - \mathbf{HP}^1 \cdot \mathbf{CaP}^2) \cdot M_3 \\ & (E_6 - \mathbf{HP}^2 \cdot \mathbf{CaP}^2) \cdot M_2 \\ & (E_7 - \mathbf{HP}^3 \cdot \mathbf{CaP}^2) \cdot M_1 \quad (E'_7 - \mathbf{HP}^3 \cdot \mathbf{CaP}^2) \cdot M_1 \end{aligned}$$

Lemma 8 implies that $\alpha_8 = s_8(p)$ and $\beta_8 = s_{4,4}(p)$ vanish on all of them. By Propositions 11 & 12:

$$\begin{vmatrix} \alpha_8(E_8 - \mathbf{CaP}^2 \cdot \mathbf{HP}^4) & \alpha_8(E'_8 - \mathbf{CaP}^2 \cdot \mathbf{HP}^3 \cdot \mathbf{HP}^1) \\ \beta_8(E_8 - \mathbf{CaP}^2 \cdot \mathbf{HP}^4) & \beta_8(E'_8 - \mathbf{CaP}^2 \cdot \mathbf{HP}^3 \cdot \mathbf{HP}^1) \end{vmatrix} = \begin{vmatrix} -3400 & -15776 \\ 2932 & 11024 \end{vmatrix} \neq 0 \quad \square$$

The final step is to show that the point of intersection $\mathbf{CaP}^2 = \mathbf{HP}^3 = \mathbf{HP}^2 = 0$ corresponds to the \hat{A} genus. This follows from Proposition 9 together with the fact that the \hat{A} genus is the point $[\delta, \varepsilon] = [-\frac{1}{8}, 0]$ of $\text{Proj } \mathbf{Q}[\delta, \varepsilon]$. Alternatively it follows since $\mathbf{HP}^2, \mathbf{HP}^3, \mathbf{CaP}^2$ are homogeneous spaces and hence admit metrics of positive scalar curvature and therefore have $\hat{A} = 0$ by Lichnerowicz's theorem [Lic63].

5. Proof of Theorem 3

Since \mathbf{CaP}^2 and \mathbf{HP}^n are spin manifolds so are the total spaces E_n and E'_n . Therefore, since the forgetful map $\text{MSpin}_* \rightarrow \text{MSO}_*$ is an isomorphism tensor \mathbf{Q} , the proofs of Theorems 1 & 2 already prove Theorem 3.

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